A new bound for the large sieve inequality with power moduli

Karin Halupczok

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Abstract

We give a new bound for the large sieve inequality with power moduli q^k that is uniform in k. The proof uses a new theorem due to T. Wooley from his work on efficient congruencing.

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1 Introduction

Let $\{v_n\}$ denote a sequence of complex numbers, $M, N, k \in \mathbb{N}$, and let Q be a real number ≥ 1. We write $e(\alpha) := \exp(2\pi i\alpha)$ for $\alpha \in \mathbb{R}$.

The large sieve inequality with power moduli aims to give upper bounds for the sum

$$\Sigma_{Q,N,k} := \sum_{q\leq Q} \sum_{\substack{1\leq a\leq q^k \\ \gcd(a,q)=1}} \Big| \sum_{M<n\leq M+N} v_n e\Big(\frac{a}{q^k} n\Big)\Big|^2.$$

It is known that an application of the standard large sieve inequality gives the upper bounds

$$\Sigma_{Q,N,k} \ll_k (N + Q^{2k})|v|^2 \quad\text{and}\quad \Sigma_{Q,N,k} \ll_k (QN + Q^{k+1})|v|^2, \tag{1}$$

where $|v|^2 := \sum_{M<n\leq M+N} |v_n|^2$, and it is conjectured by L. Zhao in [5] that the upper bound

$$\Sigma_{Q,N,k} \ll_{k,\varepsilon} |v|^2 (N + Q^{k+1})(NQ)^\varepsilon \tag{2}$$

should hold.

The bounds (1) verify the conjecture for $Q \leq N^{1/(2k)}$ and $Q \geq N^{1/k}$, so the problem is to prove it in the range

$$N^{1/(2k)} \leq Q \leq N^{1/k} \Leftrightarrow Q^k \leq N \leq Q^{2k}.$$

Especially the cases for small k, namely $k = 2, 3$ are of interest and were considered in the papers [1],[2] and [5]. In this paper we investigate the problem uniform in k. The following nontrivial bounds are known in this case.

L. Zhao showed in [5] the bound

$$\Sigma_{Q,N,k} \ll_{k,\varepsilon} |v|^2(Q^{k+1} + (NQ^{1-1/\kappa} + N^{1-1/\kappa}Q^{1+k/\kappa})N^{\varepsilon}), \tag{3}$$

where $\kappa := 2^{k-1}$.

In [2], it was shown by S. Baier and L. Zhao that

$$\Sigma_{Q,N,k} \ll_{k,\varepsilon} |v|^2(Q^{k+1} + N + N^{1/2+\varepsilon}Q^k)(\log\log 10NQ)^{k+1} \tag{4}$$

holds, which improves Zhao's bound (3) for $Q \ll N^{(\kappa-2)/(2(k-1)\kappa-2k)-\varepsilon}$.

In this paper we prove the following result:

Theorem 1. *Let $\delta := (2k(k-1))^{-1}$. Then we have the bound*

$$\Sigma_{Q,N,k} \ll_{k,\varepsilon} |v|^2(NQ)^{\varepsilon}(Q^{k+1} + Q^{1-\delta}N + Q^{1+k\delta}N^{1-\delta}).$$

This bound improves the bound (3) for all k sufficiently large, and the bound (4) for $Q^k \leq N \leq Q^{2k-2+2\delta}$ and all $k \geq 3$, but it does not confirm any case of Zhao's conjecture (2), too. Further, the result is not sufficient to give an improvement of the bound in [2] for $k = 3$, but comes near to it.

Notation. In the following, we suppress the dependence of the implicit constants on k or ε in our estimates and write simply $\ll$ for $\ll_{k,\varepsilon}$. The small value $\varepsilon > 0$ may depend on k and may change its value during calculation. The symbol $\|\alpha\|$ means the distance of α to the nearest integer, and by $\{\alpha\} := \alpha - [\alpha]$ we denote the fractional part of α, and by $[\alpha]$ the largest integer smaller or equal to α.

2 Lemmas

We make use of the following version of the large sieve inequality.

Lemma 1. *Let S denote a finite set of positive integers, $M, N \in \mathbb{Z}$ and let $\{v_n\}$ be a complex sequence. Further let*

$$\mathcal{F} := \{(a,q) \in \mathbb{Z}^2;\ q \in S,\ 0 < a < q,\ \gcd(a,q) = 1\}.$$

Then

$$\sum_{(a,q)\in\mathcal{F}} \Big| \sum_{M<n\le M+N} v_n e\Big(\frac{a}{q}n\Big)\Big|^2 \le \sum_{M<n\le M+N} |v_n|^2 \Big(4\sum_{q\in S} q + \max_{(b,r)\in\mathcal{F}} \int_{1/N}^{1/2} \#\mathcal{F}_{b,r}(x)\frac{dx}{x^2}\Big), \quad (5)$$

where

$$\mathcal{F}_{b,r}(x) := \Big\{(a,q) \in \mathcal{F};\ \Big|\frac{a}{q} - \frac{b}{r}\Big| \le x\Big\}.$$

Proof: We use Halasz-Montgomery's inequality

$$\sum_{r\le R} |\langle v, \varphi_r\rangle|^2 \le |v|^2 \cdot \max_{r\le R} \sum_{s\le R} |\langle \varphi_r, \varphi_s\rangle|$$

that holds for any sequence $\{\varphi_r\}$ of vectors of $\mathbb{C}^N$, and where $|v|^2 = \langle v, v\rangle$, and $\langle \cdot, \cdot\rangle$ is the standard scalar product on $\mathbb{C}^N$.

So the left hand side of (5) is

$$\begin{aligned}
&\sum_{(a,q)\in\mathcal{F}} \Big| \sum_{M<n\le M+N} v_n e\Big(\frac{a}{q}n\Big)\Big|^2 \\
&\le |v|^2 \max_{(b,r)\in\mathcal{F}} \sum_{(a,q)\in\mathcal{F}} \Big| \sum_{M<n\le M+N} e\Big(\frac{a}{q}n\Big) e\Big(-\frac{b}{r}n\Big)\Big| \\
&\le |v|^2 \max_{(b,r)\in\mathcal{F}} \sum_{(a,q)\in\mathcal{F}} \min\Big(N, \Big\|\frac{a}{q} - \frac{b}{r}\Big\|^{-1}\Big).
\end{aligned}$$

Now we have to estimate

$$\max_{(b,r)\in\mathcal{F}} \sum_{(a,q)\in\mathcal{F}} \min\Big(N, \Big\|\frac{a}{q} - \frac{b}{r}\Big\|^{-1}\Big). \quad (6)$$

For this, fix $(b, r) \in \mathcal{F}$. For $\Delta > 0$ write

$$P(\Delta) := \#\mathcal{F}_{b,r}(\Delta).$$

Let $\Delta_0 := \frac{1}{N}$ and for $L \in \mathbb{N}$ let $h := (\frac{1}{2} - \frac{1}{N})L^{-1}$. Now let $\Delta_i := \frac{1}{N} + hi$, so $\Delta_L = \frac{1}{2}$. Since $\|\alpha\| = \min\{|\alpha|, 1 - |\alpha|\}$ for $-1 < \alpha < 1$, we have

$$\begin{aligned}
&\sum_{(a,q)\in\mathcal{F}} \min\Big(N, \Big\|\frac{a}{q} - \frac{b}{r}\Big\|^{-1}\Big) \\
&\leq 2NP\Big(\frac{1}{N}\Big) + 2 \sum_{0\leq i<L} \sum_{\substack{(a,q)\in\mathcal{F} \\ \Delta_i<|a/q-b/r|\leq\Delta_{i+1}}} \frac{1}{\Delta_i} \\
&= 2NP\Big(\frac{1}{N}\Big) + 2 \sum_{0\leq i<L} \frac{1}{\Delta_i}(P(\Delta_{i+1}) - P(\Delta_i)) \\
&= 2 \sum_{0\leq i<L} \Big(\frac{1}{\Delta_i} - \frac{1}{\Delta_{i+1}}\Big)P(\Delta_{i+1}) + \frac{2}{\Delta_L}P(\Delta_L).
\end{aligned}$$

The last summand is $\leq 4\sum_{q\in S} q$, and the sum over i approximates the Riemann-Stieltjes-integral

$$\int_{1/N}^{1/2} P(x)dg(x) \text{ with } g(x) = -\frac{1}{x}, \tag{7}$$

if $L \to \infty$. Therefore the sum over $(a, q) \in \mathcal{F}$ in (6) is at most as large as the integral (7), plus $4\sum_{q\in S} q$.

Since g is continuously differentiable on $[\frac{1}{N}, \frac{1}{2}]$ and since P is Riemann-integrable, the integral (7) equals

$$\int_{1/N}^{1/2} P(x)g'(x)dx = \int_{1/N}^{1/2} P(x)\frac{dx}{x^2} = \int_{1/N}^{1/2} \#\mathcal{F}_{b,r}(x)\frac{dx}{x^2}.$$

This was to be shown. □

Further we use the following estimate for the exponential sum occurring in the proof of Theorem 1.

Lemma 2. *Let $f(x) := \alpha x^k \in \mathbb{R}[x]$ be a monomial of degree $k \geq 2$, and $S_Q := \sum_{Q<q\leq 2Q} e(f(q))$, $\delta := (2k(k-1))^{-1}$. Then*

$$S_Q \ll Q^{1+\varepsilon}\Big(Q^{-1} + Q^{-k}\sum_{1\leq v\leq Q}\min(Q^k v^{-1}, \|v\alpha\|^{-1})\Big)^{\delta}.$$

Proof:

Suppose that $a, q \in \mathbb{Z}$ with $(a,q) = 1$ and $|q\alpha - a| \leq q^{-1}$.

We apply Theorem 1.5 in T. Wooley's article [5] on efficient congruencing and obtain

$$S_Q \ll Q^{1+\varepsilon}(q^{-1} + Q^{-1} + qQ^{-k})^{\delta}.$$

By a standard transference principle (see Ex. 2 of section 2.8 in Vaughan's book [3]), this implies that

$$S_Q \ll Q^{1+\varepsilon}\Big((v + Q^k|v\alpha - u|)^{-1} + Q^{-1} + (v + Q^k|v\alpha - u|)Q^{-k}\Big)^{\delta} \qquad (8)$$

for any integers $u, v \in \mathbb{Z}$ with $(u,v) = 1$ and $|v\alpha - u| \leq v^{-1}$.

Now by Dirichlet's Approximation Theorem, there exist such integers u, v with $1 \leq v \leq Q^{k-1}$ and $|v\alpha - u| \leq Q^{1-k}$, for these

$$(v + Q^k|v\alpha - u|)Q^{-k} \ll (Q^{k-1} + Q)Q^{-k} \ll Q^{-1}$$

holds. Further we get

$$(v + Q^k|v\alpha - u|)^{-1} \ll Q^{-k}\min(Q^k v^{-1}, |v\alpha - u|^{-1}).$$

Now if $v > Q$, this expression is again $\ll Q^{-1}$. If otherwise $1 \leq v \leq Q$, it is bounded by

$$Q^{-k}\sum_{1\leq v\leq Q}\min(Q^k v^{-1}, \|v\alpha\|^{-1}),$$

since $|v\alpha - u| \geq \|v\alpha\|$.

Hence, these estimates included in (8) show the assertion.

$\square$

Lemma 3. *Let $X, Y, \alpha \in \mathbb{R}$, $X, Y \geq 1$, and $a, q \in \mathbb{Z}$, $\gcd(a,q) = 1$, with $|q\alpha - a| \leq q^{-1}$. Then*

$$\sum_{v\leq X}\min\left(XYv^{-1}, \|\alpha v\|^{-1}\right) \ll XY(q^{-1} + Y^{-1} + q(XY)^{-1})\log(2Xq).$$

This is Lemma 2.2 of [3]. $\square$

3 Proof of Theorem 1

Let $k \in \mathbb{N}$ with $k \geq 2$, let $Q \geq 1$ and assume that the integer N is in the range $Q^k \leq N \leq Q^{2k}$.

We apply Lemma 1 with

$$\mathcal{F} := \{(a, q^k) \in \mathbb{Z}^2;\ Q < q \leq 2Q,\ 0 < a < q^k,\ \gcd(a, q) = 1\},$$

which shows that

$$\Sigma_{Q,N,k} \ll |v|^2 Q^{\varepsilon}\Big(Q^{k+1} + \max_{(b,r^k)\in\mathcal{F}} \int_{1/N}^{1/2} \#\mathcal{F}_{b,r^k}(x)\frac{dx}{x^2}\Big),$$

since we have the admissible error $\sum_{q\leq Q} q^k \ll Q^{k+1}$.

Now we aim to give an estimate for

$$\max_{(b,r^k)\in\mathcal{F}} \int_{1/N}^{1/2} \#\mathcal{F}_{b,r^k}(x)\frac{dx}{x^2}$$

The integrand counts for fixed $(b, r^k) \in \mathcal{F}$ all $(a, q^k) \in \mathcal{F}$ with

$$\left|\frac{a}{q^k} - \frac{b}{r^k}\right| \leq x.$$

So for fixed $Q < q \leq 2Q$, we count every a with

$$\frac{|ar^k - bq^k|}{2r^k Q^k x} \leq \frac{1}{2}.$$

Now we use the Fourier analytic method from the papers [1],[2] and [5] by Baier and Zhao. For this, consider the function

$$\phi(x) := \Big(\frac{\sin \pi x}{2x}\Big)^2, \qquad\qquad \phi(0) := \lim_{x\to 0}\phi(x) = \frac{\pi^2}{4}.$$

Then $\phi(x) \geq 1$ for $|x| \leq 1/2$, and the Fourier transform of ϕ is

$$\hat{\phi}(s) = \frac{\pi^2}{4}\max\{1 - |s|, 0\}.$$

For fixed q, we get for the number of corresponding a the estimate

$$\sum_{a,(a,q)\in\mathcal{A}_{b,r}(x)} 1 \leq \sum_{a\in\mathbb{Z}} \phi\Big(\frac{ar^k - bq^k}{2r^k Q^k x}\Big) = \sum_{a\in\mathbb{Z}} \int_{-\infty}^{\infty} \phi\Big(\frac{sr^k - bq^k}{2r^k Q^k x}\Big)e(as)ds,$$

where we applied in the last step Poisson's summation formula. Summing up over q and a linear transformation gives

$$\sum_{Q<q\leq 2Q} \sum_{a\in\mathbb{Z}} \int_{-\infty}^{\infty} \phi(v) e\Big(ab\frac{q^k}{r^k}\Big) e(2Q^k xav) 2Q^k x dv$$
$$= \sum_{|a|\leq B} \hat{\phi}\Big(\frac{a}{B}\Big) B^{-1} \sum_{Q<q\leq 2Q} e\Big(ab\frac{q^k}{r^k}\Big),$$

where we have set $B := (2Q^k x)^{-1}$, and we may assume w.l.o.g. that $B \geq 1$.

We separate the summand with $a = 0$ and get

$$\ll Q^{k+1}x + B^{-1} \sum_{1\leq a\leq B} \Big| \sum_{Q<q\leq 2Q} e\Big(\frac{abq^k}{r^k}\Big)\Big|.$$

The separated term $Q^{k+1}x$ leads again to the admissible contribution

$$\int_{1/N}^{1/2} Q^{k+1}\frac{dx}{x} \ll Q^{k+1+\varepsilon}.$$

Consider the monomial $f(q) := \frac{ab}{r^k}q^k$ of degree k in q and coefficient $\alpha := \frac{ab}{r^k} \neq 0$. It remains to give a good upper bound for the expression

$$\int_{1/N}^{1/2} B^{-1} \sum_{1\leq a\leq B} \Big| \sum_{Q\leq q<2Q} e(f(q))\Big| \frac{dx}{x^2}. \tag{9}$$

Denote by S_Q the occurring exponential sum

$$S_Q := \sum_{Q<q\leq 2Q} e(f(q)).$$

By Lemma 2, we have

$$S_Q \ll Q^{1+\varepsilon}\Big(Q^{-1} + Q^{-k} \sum_{1\leq v\leq Q} \min(Q^k v^{-1}, \|v\alpha\|^{-1})\Big)^{\delta}.$$

The summand Q^{-1} in big parantheses provides already the contribution

$$\int_{1/N}^{1/2} Q^{1-\delta+\varepsilon}\frac{dx}{x^2} \ll Q^{1-\delta+\varepsilon}N \tag{10}$$

to (9), and it remains to consider the term with the sum over v.

We estimate its contribution to S_Q as follows using Hölder's inequality and Lemma 3. We have

$$\begin{aligned}
&Q^{1+\varepsilon-k\delta}\sum_{a\le B}\Big(\sum_{v\le Q}\min\Big(Q^k v^{-1}, \Big\|\frac{ab}{r^k}v\Big\|^{-1}\Big)\Big)^{\delta}\\
\ll&Q^{1+\varepsilon-k\delta}B^{1-\delta}\Big(\sum_{\ell\le BQ} d(\ell)\min\Big(BQ^k\ell^{-1}, \Big\|\frac{b}{r^k}\ell\Big\|^{-1}\Big)\Big)^{\delta}\\
\ll&Q^{1+\varepsilon-k\delta}B^{1-\delta}\Big((BQ^k)^{1+\varepsilon}(r^{-k}+Q^{1-k}+r^k(BQ^k)^{-1})\Big)^{\delta}\\
\ll&BQ^{1+\varepsilon}(Q^{1-k}+B^{-1})^{\delta}.
\end{aligned}$$

The contribution to (9) becomes

$$\begin{aligned}
&\ll Q^{1+\varepsilon+(1-k)\delta}N + Q^{1+\varepsilon}\int_{1/N}^{1/2} B^{-\delta}\frac{dx}{x^2}\\
&\ll Q^{1-(k-1)\delta+\varepsilon}N + Q^{1+\varepsilon}\int_{1/N}^{1/2} Q^{k\delta}x^{\delta}\frac{dx}{x^2}\\
&\ll Q^{1-(k-1)\delta+\varepsilon}N + Q^{1+k\delta+\varepsilon}N^{1-\delta}.
\end{aligned}$$

The first term can be estimated by the bound (10), since $k \ge 2$. We obtain the stated bound of Theorem 1. □

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